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Mathematics in Physics Education: Scanning Historical Evolution of the Differential to Find a More Appropriate Model for Teaching Differential Calculus in Physics

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Abstract. Despite its frequent use, there is little understanding of the concept of differential among upper high school and undergraduate students of physics. As a first step to identify the origin of this situation and to revert it, we have done a historic and epistemological study aimed at clarifying the role and the meaning of the differential in physics and at improving curricular and teaching models in the sense of Gilbert et al. (Gilbert J.K., Boulter C., & Rutherford, M.: 1998a, *International Journal of Science Education* **20**(1), 83–97, Gilbert J.K., Boulter C., & Rutherford, M.: 1998b, *International Journal of Science Education* **20**(2), 187–203). We describe the contributions of Leibniz and Cauchy and stress their shortcomings, which are overcome by the alternative definition proposed by the French mathematician Fréchet, dating from early 20th century. As a result of this study, we answer to some fundamental questions related to a proper understanding of the differential in physics education (for 17–19 years old students).

1. Introduction and Approach to the Problem

Differential calculus is first used in the last years of high school, and it is present in the majority of physics topics at university. Differential calculus is necessary for the analysis of physical problems with a certain degree of complexity, ones that are closer to reality than those dealt with in elementary courses: the history of science shows how the invention of the differential calculus meant a step forward in the type and complexity of the problems that could be solved (Edwards 1937; Aleksandrov et al. 1956; Kline 1972). Differential calculus has been the main quantitative tool to progress in the understanding of scientific problems for the last three centuries, and without it physics and modern technology would not exist (Kleiner 2001).

In contrast, the results derived from various studies, mainly in the field of mathematics teaching, have revealed considerable shortcomings among students, and even among teachers, as regards the basic concepts of calculus (Orton 1983a, b; Tall 1985a, 1992; Azcárate 1990; Ferrini-Mundy & Geuther 1991; Schneider 1991; Ferrini-Mundy & Gaudard 1992; Thompson 1994; Thompson & Thompson 1996; Porter & Masingila 2000), and to be more precise, in relation to the concept of differential (Tall 1985c; Alibert et al. 1987; Artigue & Viennot 1987). For our part, we have shown that only a small fraction of physics students in pre-university courses, and in the first courses of scientific-technical degrees, use differential calculus in physics fully understanding what they are doing. There is a strong uneasiness among physics and chemistry teachers in high school, as at this level teachers are unable to overcome these shortcomings (López-Gay et al. 2001, 2002; Martínez-Torregrosa & López-Gay, 2005). We carried out a survey and concluded that, out of 103 teachers in high school who actively take part in training courses, 88% recognise that 'the teachers themselves do not master differential calculus sufficiently when solving problems in new contexts', and only 22% are absolutely sure that they know when and why to use differential calculus in physics.

This poor understanding may greatly influence student and teachers' expectations and attitudes. Teachers have low expectations that students of physics in their pre-university course understand the use of differential calculus; and 65% of pre-university students ($n = 108$), 77% of students of physics in their first year of scientific-technical degrees ($n = 116$) and 64% in their second year ($n = 63$) recognise that 'teachers apply differential calculus because it is essential for the development of the topic, however, they do not expect us to understand it'. Students therefore consider differential calculus to be an obstacle which generates insecurity and anxiety (Tall 1985a; Lavaly 1990; Aghadiuno 1992; Martin & Coleman 1994; Monk 1994). We are therefore dealing with a situation that may worsen the attitudes towards physics and mathematics, about which so much interest and concern has arisen throughout Western societies (Osborne et al. 2003).

Many studies agree that such difficulties concerning differential calculus arise mainly from inappropriate algorithmic teaching methods (Orton 1983a, b; Artigue & Viennot 1987; Ferrini-Mundy & Geuther 1991; Nagy et al. 1991; Schneider 1991; Ferrini-Mundy & Gaudard 1992; Thompson 1994; Thompson & Thompson 1994). Porter and Masingila (2000) state that many students are unaware what concepts underlie these procedures. These students believe that mathematics implies only specific operations with meaningless symbols and they learn mathematics in a mechanical way (as the teacher did, in his turn). We carried out several interviews and when we asked teachers and teacher trainees about the meaning of derivatives and integrals, the most frequent answer was: 'I know how to calculate them, but I don't know what they mean'.

Sometimes the algorithmic approach is justified as an appropriate starting point. Nevertheless, some researchers warn that technique-oriented courses in calculus hinder the subsequent conceptual understanding (Ferrini-Mundy & Gaudard 1992; Porter & Masingila 2000). At most, they provide students with an 'instrumental understanding', useful to reproduce manipulative routines but insufficient to solve problems if unaccompanied by conceptual understanding (White & Mitchelmore 1996). The calculus reform movement started in the USA states that the algorithmic approach brought about the critical situation calculus teaching is going through in that country, and puts forward a conceptual approach, based on fully understanding what is done and why. This approach has inspired other work concerning the concepts of derivative and integral, and of prior concepts such as those of limit or function (Orton 1983a, b; Tall 1985b, 1986; Azcárate 1990; Ferrini-Mundy & Geuther 1991; Williams 1991; Ferrini-Mundy & Gaudard 1992; Schneider 1992; Thompson 1994; Turégano 1998; Gravemeijer & Dorman 1999). The main point is to acknowledge that the key ideas should prevail over the hundreds of formulae and techniques (Kleiner 2001), bearing in mind that calculus implies algorithms and concepts, and that students will have to deal with both. This point constitutes a general recommendation of the Principles and Standards for School Mathematics, which is to restore the proportion between the mastery of algorithms and conceptual understanding, in favour of the second aspect (NCTM 2000).

We agree with the need to improve the teaching of differential calculus in physics. This change is to be based on the full comprehension of what is done and why it is done. Nevertheless, in contrast to most studies mentioned above, our concern is not strictly mathematical but is related with its physical applications. For this reason, our study is centred on the differential, which is usually pushed into the background when teaching mathematics. However, the differential is frequently used for reasoning and *mathematisation* in physical contexts during the last year of high school and first university courses of technical careers, either for theoretical developments or for solving problems in which expressions like $\mathrm{d}x = v \cdot \mathrm{d}t$, $\mathrm{d}p = F \cdot \mathrm{d}t$, $\mathrm{d}W = F \cdot \mathrm{d}x$, $\mathrm{d}V = -E_r \cdot \mathrm{d}r$, $\mathrm{d}F = B \cdot I \cdot \mathrm{sen}\theta \cdot \mathrm{d}l$, $\mathrm{d}N = -\lambda \cdot N \cdot \mathrm{d}t$, etc., are already present in the initial reasoning.

In order to understand the causes for the usual shortcomings in teaching and in learning practice, and to be able to introduce proposals to overcome them, it is first necessary to explain the conceptual meaning of what is being done when differential calculus is used in physics. We carried out a historical study of the evolution of differential calculus, concentrating on the concept of differential, pointing out the obstacles that had to be overcome, the questions which needed to be answered and the changes that allowed its progress to reach the current conceptions (Martínez-Torregrosa et al. 1994). What we have learnt from this historical and epistemological study has allowed us to accurately identify what exactly an appropriate understanding of the differential in physics involves, as well as to analyse and understand the origin of the shortcomings that are found nowadays. This work has enabled us to draw a teaching program for the classroom that does not reproduce the historical development, neither does follow literally the last concept of differential that has been mentioned, but it takes into account the evolution of the ideas concerning the concepts, in order to bring about conceptual transparence and to reconcile the physical and mathematical meaning according to the level of our students. In this article, we focus on presenting the explanation of the concept of the differential and some of the obstacle-ideas that had to be overcome in order to arrive at its current conception.

2. Historical Conceptions of the Concept of Differential

The enormous success that calculus had in the 17th and 18th centuries was not accompanied by a clear understanding of what was being done. In contrast with the image of exactness and rigor that mathematical texts provide from the first page on, this lack of comprehension provoked a gradual accumulation of contradictions and deficiencies. In referring to this time, Eves (1981) states: 'Attracted by the powerful applicability of the subject, and lacking a real understanding of the foundations upon which the subject must rest, mathematicians manipulated analytical processes in an almost blind manner, often being guided only by a naïve intuition of what was felt must be valid' (p. 134).

Many of these shortcomings directly influenced the meaning and importance of the differential. Following the categorisation of Alibert et al. (1987), we have identified two historical conceptions which are representative of the evolution of differential: Leibniz's differential and Cauchy's differential.

2.1. LEIBNIZ'S DIFFERENTIAL

Infinitely small quantities ('evanescent divisible quantities', according to Newton; and 'incipient quantities', 'not yet formed', according to Leibniz) are considered principal elements for the creation of calculus, but at the same time, its weakest point and a target for criticism. Those quantities were considered at the beginning like fixed quantities with the smallest value possible, but never zero; however, they were later considered as quantities that could be as small as required.

Leibniz and his followers (the Bernouilli brothers, Marqués de l'Hôpital, Euler...) whose notation and language became established in differential calculus, referred to the differential of a magnitude (dy) as the infinitesimal increment of that magnitude (y), (its 'moment', according to Newton). If a macroscopic value could be attributed to dy, it would not coincide with Δy. However, as only infinitely small values were attributed, dy was identified with Δy with no error. Although the differential of position (dx), in macroscopic terms did not tally with any displacement, it could be identified with the displacement occurring in an infinitely small time interval (dt).

The differential played a key role in the structure of calculus. It was used to substitute the increment to calculate the derivative (defined as the quotient of very small increments) and the integral (defined as the addition of very small infinite increments). The following examples illustrate the original use Newton and Leibniz applied in their calculations and reasoning, which were presented in geometrical form.

- To calculate the derivative of $y = x^2$ an infinitesimal variation dx will produce another infinitesimal variation dy: $y + \mathrm{d}y = (x + \mathrm{d}x)^2 = x^2 + 2x \cdot \mathrm{d}x + \mathrm{d}x^2$; therefore: $\mathrm{d}y = 2x \cdot \mathrm{d}x + \mathrm{d}x^2$. Both elements were divided by dx: $\mathrm{d}y/\mathrm{d}z = 2x + \mathrm{d}x$; only then the infinitesimal addenda were disregarded, resulting in: $\mathrm{d}y/\mathrm{d}x = 2x$.
- To confirm the inverse relationship between derivatives and the calculation of areas $A(x)$ under curves $y(x)$, it was considered that an infinitesimal variation dx produced an infinitesimal variation dA, which could be approximated to a rectangle of height $y(x)$ and width dx, therefore: $\mathrm{d}A = y \cdot \mathrm{d}x$. To obtain: $\mathrm{d}A/\mathrm{d}x = y$, both elements were divided by dx.

They considered dx to be an infinitesimal displacement produced in an infinitesimal time interval dt (although dx never coincides with Δx, it could replace it in such a short time interval) and the instantaneous velocity would be the quotient of these infinitesimal quantities.

As it shows, the use of infinitesimals entails certain advantages since an equal sign was written when, in fact, it was only an approximation in the case of finite increments. This, according to Freudenthal (1973), may be distressing for mathematicians nowadays. Furthermore, these terms could be neglected when appropriate (Marqués de L'Hôpital summarised this in the equation: $y + \mathrm{d}y = y$). However, the use of infinitesimals also entails serious doubts and harsh criticism which could be summed up as follows:

- How can the deletion of some terms be justified? The fact of simply saying that the neglected quantities were zero at the end, without clarifying why at the beginning they were not zero, seemed to violate the Principle of Identity, according to which there is no intermediate stage between equality and difference (even for very small differences) for two mathematical entities. George Berkeley categorically came to the conclusion that: 'It is impossible to achieve real propositions from false principles' (quoted by Rossi 1997, p. 204). In the same sense, D'Alembert disagrees: 'A quantity is either 'something' or 'nothing at all'; if it is 'something', it still has not disappeared; if it is 'nothing', it has literally vanished. The assumption that there is an intermediate phase between both of them is an illusion' (quoted by Romero & Serrano 1994).

- It was also said that they were not zero but could be disregarded against much larger quantities. In such case, is it possible to obtain an exact result by disregarding terms that are not zero?
- Nieuwentijdt, a Dutch physicist and mathematician, wondered: how can the sum of infinitesimals that can be neglected lead to a finite result? (Kline 1972, p. 509)
- What criteria should be used to move from an approximate expression in terms of increments to an exact one in terms of differentials? Could it be applied to any other expression? For instance, to calculate the surface area of a sphere, ΔA can be estimated by adding infinitesimal cylindrical surfaces or infinitesimal surfaces of truncated cones. These alternatives make it difficult to choose the correct expression for the differential (Artigue & Viennot 1987), which lead to different results. The situation is even worse in most physical contexts in which there are many introductory expressions that relate very small increments in an approximate way. The intuitive belief that the sum of infinite infinitely-small quantities would produce the desired large quantity, disregarding the 'shape' of the quantities, failed in many occasions, leading to absurd results (Schneider 1991).

Leibniz and Newton were not able to answer these criticisms and objections clearly, mainly due to lacking a clear definition of the concept of a limit. In his latest studies, Newton tried to avoid the use of infinitesimals: 'in mathematics, not even the slightest errors should be disregarded', he said (quoted by Kline, 1972, p. 480). But it was a theoretical attempt to avoid contradictions; as Berkeley reported: 'after all, it is essential to refer back to the idea of evanescent increments' (quoted by Edwards 1937, p. 294).

Leibniz admits that he 'does not believe in truly infinite or infinitesimal magnitudes' (Kline 1972, p. 511); nevertheless, he defends their use for practical reasons by considering the symbols as 'useful fictions to abbreviate and to speak universally' (Edwards 1973, p. 264). Leibniz retorted to Nieuwentijdt's criticism by saying: 'infinite and infinitely small quantities can be used as a tool, in the same way as algebraists satisfactorily used imaginary roots' (Kline 1972, p. 509).

This historical briefing shows that the definition of differential is incomplete, not only due to a lack of arguments explaining how and why calculus works, but also because it leads to wrong results. The wrong belief that any approximate expression for the increment can be considered exact for infinitely small intervals, that is, when it turns into a differential expression, made it difficult to understand why in certain cases the algorithm failed. The importance the application of calculus gained when solving many problems, together with the lack of understanding and justification of what was being done, stamped a mechanical and repetitive strategy on it, which was more concerned with algorithms than with their meaning.

The results shown in other publications point out that this historical conception and the mechanical attitudes are also present in physics teaching (Artigue 1986; López-Gay et.al. 2001, 2002; Martínez-Torregrosa & López-Gay, 2005). This situation may be expected in the process of creating and fixing new knowledge, but not in nowadays teaching practice.

The situation can be illustrated by the following extract from an interview carried out with a bright student in his pre-university course:

Juan: Every time we use differentials, my teacher says: 'In order to analyse this curve we are going to take the straight lines as small as we please...' (...) I don't understand it... In truth, I know how to calculate integrals, but I haven't actually understood the differentials that occur, I see them in writing but I don't know what they are...and, why am I going to bother asking, since they are going to tell me: 'these are the little pieces...'.

2.2. CAUCHY'S DIFFERENTIAL

Lagrange, already aware in 1784 of the inaccuracies and ambiguities in the use of the infinite and infinitesimals, announced in the Academy of Berlin a competition to replace such ideas while maintaining simplicity in the reasoning. In the face of a lack of satisfactory answers, Lagrange put forward his own solution, a theory of analytical functions in which infinitely small quantities were disregarded in the differential calculus and the concept of derivative was considered a central concept. However, it remained restricted to a theoretical development, because Lagrange in his Analytic Mechanics referred back to the differential and to infinitely small quantities when dealing with physical situations (Laugwitz 1997a).

Cauchy disagreed with the solution put forward by Lagrange. The fact that he was concerned about the validity of the solutions to problems like those of vibrating strings and heat conduction, and interested in clarification for university teaching, enabled him to consolidate calculus on a rigorous grounding (Kleiner 2001). With a better understanding of limit and real numbers, Cauchy formulated an accurate definition of infinitesimal quantities, derivative, and integral. The differential was no longer identified with an infinitesimal increment; it was emptied of physical meaning and moved into a peripheral position within the structure of calculus.

An infinitesimal was defined as a variable whose numerical value decreases indefinitely in such a way that it has zero for its limit (Cauchy

1821, pp. 26–27), which overcomes some of the objections formulated in previous centuries. In particular:

– The application of the concept of limit to an expression produces a new mathematical object. For example, for the function $y = x^2$ the incremental quotient is $\Delta y/\Delta x = 2x + \Delta x$; its value will never be $2x$, no matter how small Δx is; however, when calculating the limit of that function when Δx tends to zero, the new object (which is not a quotient of increments, in the same way that the limit of a sequence does not have to belong to such a sequence) will be $2x$. Therefore, when there are infinitesimals in an expression, these quantities are not zero, although they have zero for their limits.
– Infinitesimals are not small quantities, but variables or functions ($f(x)$) that satisfy a condition: when x tends to zero, its value is zero. This feature does not restrict the numerical value of that variable or function.

These concepts hinge on an appropriate understanding of limit, which happened neither in Cauchy's time nor happens nowadays in the teaching of physics and mathematics (Williams 1991; Schneider 1992; Lauten et al. 1994; Cottrill et al. 1996; Sánchez & Contreras 1998). Cauchy himself was doubtful about the meaning of infinitesimals: he sometimes referred to the infinitely small 'value' of these quantities, and he even considered them very small numbers, allowing for their manipulation independently. This confusion may have forced Cauchy to dispense with the infinitesimals when teaching analysis. This reaction led to a confrontation with the physicist Petit and with the Conseil d'Instruction at the École Polytechnique, since they did not understand he omitted the infinitesimals in his theoretical classes given that they were useful to solve practical problems (Laugwitz 1997b).

The definition of the limit also provided an accurate definition of the derivative and the integral, as well as a clear procedure for computing them. The derivative was defined as the limit of a quotient of increments; the integral, which after the Fundamental Theorem was, in practice, used only for inverse operations of the derivative, owing to Cauchy regained the important role it had played during the first half of the 17th century and was interpreted as the limit of a series of additions of approximate increments.

The differential was therefore no longer essential in order to define and calculate derivatives and integrals and was placed in a peripheral position within the theoretical framework of calculus. Cauchy defined the differential as an expression involving the derivative: $\mathrm{d}f = f'(x) \cdot \mathrm{d}x$, with an arbitrary (big or small) increment $\mathrm{d}x$ of the variable and it thus became a simple formal instrument, necessary for the abbreviation of certain proofs. The differential was then detached from the ambiguity of the infinitely small quantities, but it was devoid of all physical meaning: it was just the result of multiplying the derivative by the increment of the independent variable. As Freudenthal (1973, p. 550*)* says: "Useless differentials can readily be dismissed. If dy and dx occur only in the combination dy/dx, or under the integral sign after the integrand, the question as to what dx and dy mean individually is as meaningful as to ask what the 'l', 'o', 'g' in 'log' mean" .

Although it complies with the mathematical requirements, this subordinate conception of the differential is not sufficient for physical applications, in which the differential expressions – lacking any physical meaning for Cauchy's followers – were still considered as an intuitive starting point, as an approximation to the increment and coinciding with it when it is infinitesimal. Mathematicians solved differential equations put forward by physicists, disregarding their meaning: they simply divided them by dx, to convert them into derivative equations; only then the derivative and the integral gained significance. Physicists, using Leibniz's conception, regarded the differential in an intuitive way and could not combine physical sense with rigour. Since mathematicians considered differentials to be meaningless, they could not solve questions like these: What criteria are applied to find the differential expression for a given physical context? For instance, what argument has to be used to decide the differential expression that represents the process of absorption of a plane wave in a material medium?

The lack of physical meaning is not the only drawback in Cauchy's new analysis, but shows the difficulties of applying a purely mathematical language which is far from physical reality. Despite a precise definition of the derivative and the integral, it is difficult to identify the inverse relation between them, which was easy in Leibniz's conception. Even more obvious is the difficulty to interpret the physical meaning of the concepts and expressions in which they occur. The following dialogue heading an article of the *American Mathematical Monthly* (quoted by Cuenca 1986) serves as an example:

Student: The velocity of a car is 50 miles per hour, what does this mean?

Teacher: (...) (According to Cauchy, $\lim_{\Delta t \to 0} \frac{e_2 - e_1}{t_2 - t_1} = 50$ means that...) given $\varepsilon > 0$, a given $\delta > 0$ exists, such that if $(t_2 - t_1) < \delta$ then $\frac{e_2 - e_1}{t_2 - t_1} - 50\varepsilon$.

This answer is given when the meaning of differential is rejected, reducing the concept of instantaneous velocity (the concept of derivative in general) to its operational definition as the calculation of a limit. It clearly shows that the rigour calculus acquired in the 19th century, also divorced physics from mathematics: mathematicians used calculus disregarding any physical contexts, whereas physicists would apply it as a necessary medicine in an expedite way, lacking the precision of mathematicians (Dunn & Barbanel 2000). This divorce prevails regarding the differential: in mathematics it is a formal tool in a peripheral position; in physics it represents an important approximation and reasoning tool, a very small quantity, which is central in physics reasoning (Artigue 1986; Artigue & Viennot 1987).

It seems obvious that Cauchy's contribution does not overcome the insecurity and the mechanical use of calculus for physical applications. It is therefore essential to reconcile the physical usefulness of Leibniz's and Newton's differential expressions, with the rigour and accuracy of Cauchy's formulation. To borrow Freudenthal's description (1973, p. 553): 'It is an impossible situation that the mathematician teaches a mathematics that cannot be applied and the physicist applies a mathematics that has not been taught by the mathematician'.

3. A Reconciling Concept: Fréchet's Differential. Some Answers to Fundamental Questions for a Better Teaching

In 1911 the French mathematician Fréchet put forward a new definition of differential to overcome some of the shortcomings of Cauchy's definition when he tried to extend the field of analysis to functions of several variables and even infinite variables (Alibert et al. 1987). Fréchet defines differentiability and the differential this way (Artigue 1989, p. 34):

'A function f(x, y, z, t) admits a differential, in my sense, in point (x_0, y_0, z_0, t_0) if there is a homogeneous and linear function of the increments, let it be $A\cdot \Delta x + B\cdot\Delta y + C\cdot\Delta z + D\cdot\Delta t$, that does not differ from the increment of the function Δf, that starts from the value f(x_0, y_0, z_0, t_0), in more that an infinitely small value, in relation with the distance δ between the points (x_0, y_0, z_0, t_0) and ($x_0 + \Delta x$, $y_0 + \Delta y$, $z_0 + \Delta z$, $t_0 + \Delta t$). The differential is then, by definition, $df = A\cdot\Delta x + B\cdot\Delta y + C\cdot\Delta z + D\cdot\Delta t$.

(...) This definition is expressed by the formula: $\Delta f = df + \varepsilon\cdot\delta$, where ε goes to zero when δ goes to zero. It reminds us of the old definition, as the principal part, and it presents all its advantages, but it overcomes the objections of lack of rigor that quite correctly had been put forward to it'.

Fréchet does not require the differential to be infinitely small, but that (Δf−df) is infinitely small in relation to δ; this does not mean that (Δf−df) will always be a very small number, and even less that Δf or df are small. The requirement is for (Δf−df) to go to zero faster than δ, that is to say, that the limit of (Δf−df)/δ is zero when δ goes to zero. This then means that df is a homogeneous and linear function of the increments, and we may express Δf in the following way: $df + \varepsilon \cdot \delta$, where $\lim_{\delta \to 0} \varepsilon = 0$. Dieudonné (1960, p. 145) considers the approximation of any function by linear functions as 'the fundamental idea of calculus'.

Although Fréchet's definition has its origin in the analysis of functions of infinite variables, it is also used in some textbooks to introduce the analysis of functions of one variable (Del Castillo 1980; Hallez 1989, p. 67). For this particular case, the differential would be a linear function $df = A \cdot \Delta x$ that satisfies: $\lim_{\Delta x \to 0} \frac{\Delta f - df}{\Delta x} = 0$. As df is linear, it has a constant slope $df/\Delta x$ and, therefore: $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{\Delta x}$. The first member of this equality is the derivative of the function, if it exists, and therefore in these cases Fréchet's condition is equivalent to require that $f' = df/dx$ (for a single independent variable $\Delta x = dx$).

We believe that this concept of differential may bring about the necessary reconciliation of physical usefulness with mathematical rigor and accuracy, because its definition is as precise as Cauchy's but it has a clear meaning linked to the idea of approximation, which is so important in physics since the origin of calculus. But we do not wish to present Fréchet's definition for single-variable functions as if it was an axiom, without any justification. For this reason, "inspired" in Fréchet's concept of differential we have used his basic ideas in order to clarify the use of the differential in teaching physics, by clearly identifying the problem to be solved, the global strategy that will be used to solve it, and the meaning and the requisites that the differential must fulfill if the problem has to be solved successfully (Gras-Marti et al. 2001; López-Gay et al. 2001, 2002; Martínez Torregrosa et al. 2002). We have ended up in Fréchet's definition not as an axiom but as an instrument that allows us to solve successfully an important physical problem. We outline here the main conclusions of this clarifying process in the form of answers to basic questions:

– *When is it necessary to use the differential?* When one wishes to find the function that relates the change of two physical magnitudes Δy and Δx and that relationship is not linear ($\frac{\Delta y}{\Delta x} \neq$ constant). For example, if the acceleration of a mobile varies with time according to $a = 3t + 2$ what will Δv be from t to $t + \Delta t$? We cannot write $\Delta v = a(t) \cdot \Delta t$ because a is changing through Δt. So, how to find Δv from Δt? The reason we must use the differential is the existence of an unknown non-linear relationship between two magnitudes, and not that any of them needs to have a very small value.
– *Why the differential is useful? How can it help us to solve the previous problem?* dv is a linear estimate of the increment Δv from t to $t + \Delta t$: $dv = a(t) \cdot dt$ (where $dt = \Delta t$), and has a meaning in itself: dv is what v would change in Δt if a were constant in such an interval (with the value it has at the beginning of the interval Δt). In our example, dv for $t = 3$ s and $\Delta t = 10$ s would be $dv = (3{\cdot}3 + 2) \cdot 10 = 110$ m/s. And, in functional form, $dv(t, dt) = (3t + 2) \cdot dt$. The differential is an estimate of Δv in that interval and, of course, it is a first approximation, no the exact value of Δv.

The general strategy to find the exact value is well known: one divides Δt in N subintervals and adds up the corresponding linear estimates; in this way, one gets a better and better approximation to Δv as N increases. It is advisable, in the classroom, to calculate those estimates for 2, 3... terms and to interpret the results physically. Although none of these additions will equal Δv for a finite N, it is possible that the limit of the series (the definite integral) is equal, exactly, to Δv. One may show that this will happen if the linear function that one chooses in order to make the estimates satisfies Fréchet's condition; for the usual kind of functions that appear in physics courses this is equivalent to the condition that the slope of the differential coincides with the derivative (for more details see Gras-Marti et al. 2001; López-Gay et al. 2001, 2002; Martínez Torregrosa et al. 2002). In general we may state that the relationship $y' = \mathrm{d}y/\mathrm{d}x$ is a real quotient and it expresses the condition that the differential must satisfy so that the definite integral yields an exact result for Δy (in the way of a function). In brief, if we know the differential of a function, $\mathrm{d}y = A(x) \cdot \mathrm{d}x$, the primitive function of $A(x)$ will be $y(x)$ (plus a constant). In the previous example, $v(t) = (3/2)t^2 + 2t + C$, and, therefore, $\Delta v = v(13) - v(3) = 260$ m/s.

– *What criterion must be follow in order to write the differential expression of a physical magnitude?* When we know the functional relation among the magnitudes Δx and Δy in the case of a linear behavior starting from a certain value of x, it is simple to determine the differential expression. This was the case in the previous example: we are *sure* that if starting from t the acceleration remains constant, with the value $a(t)$, for an Δt, then $\Delta v = a(t) \cdot \Delta t$. The tangent's slope to $v = f(t)$ at t, is $a(t)$, then $\mathrm{d}v = a(t) \cdot \mathrm{d}t$. The same holds for $\mathrm{d}x = v(t) \cdot \mathrm{d}t$, $\mathrm{d}p = F(t) \cdot \mathrm{d}t$, or $\mathrm{d}V = -E_r(r) \cdot \mathrm{d}r$.

But in most physical problems the relationship between Δx and Δy is not known, and even a hypothetical linear behavior cannot be assumed. In those cases the expression $\mathrm{d}y = A(x) \cdot \mathrm{d}x$ must be put forward by way of an assumption, according to the analysis of the physical situation at hand. Many expressions might be 'good candidates' from the mathematical and physical standpoint, but since the differential expression is now dealing with an unknown situation (we cannot measure $\mathrm{d}y$) we cannot be sure *a priori* about it. The physical validity of the expression we choose can only be corroborated indirectly: by way of an empirical test of the relationship between y and x that it leads to (Δy can certainly be measured) or by means of the coherence of the functional relationship we obtain with the body of knowledge in which the problem dealt with is inserted. For example, when we wish to find the relationship between the increment in the intensity ΔI of a plane wave and the path it has traveled in a certain medium, Δx, we are compelled to make reasonable assumptions. We expect that ΔI will depend on the nature of the material, as well as on Δx and the value of the intensity (the higher the intensity, the higher absorption of the material, for a given Δx). However, the expression $\Delta I = -\alpha \cdot I \cdot \Delta x$ cannot be written, as I varies with x. A possible linear estimate of ΔI, when the wave travels from x to $x + \Delta x$, could be: $\mathrm{d}I = -\alpha \cdot I(x) \cdot \mathrm{d}x$. Other possible expressions would be: $\mathrm{d}I = -\alpha \cdot I^2(x) \cdot \mathrm{d}x$, $\mathrm{d}I = -\alpha \cdot I^{1/2}(x) \cdot \mathrm{d}x$, $\mathrm{d}I = -\alpha \cdot \mathrm{d}x/x$, etc. Although mathematically one could obtain from any of the differential expressions above, via integration, *a certain* function relating I with x, only one of them will have a physical validity: the one that complied with empirical verification. The same happens when we wish to find the law of radioactive disintegration. Unfortunately, textbooks start-off from the differential which is physically correct, without making any comment about how it was selected among other possible choices.

One may recognise in these conclusions a clear reconciliation between physical sense and rigor when one uses differential calculus in physics. The clarification that we have achieved, by starting from Fréchet's concept, has allowed us to make an exhaustive analysis of current teaching practice which has confirmed the existence of important deficiencies (López-Gay et al. 2001, 2002; Martínez-Torregrosa & López-Gay, 2005); we have also been able to design sequences of activities for the teaching of physics that facilitate the use of calculus with comprehension. The good preliminary results that we are obtaining in our teaching when the need of using differential calculus arises, indicate that we are in a good path towards overcoming the deficiencies and contradictions that are usually found (Gras-Marti et al. 2001; López-Gay 2001; Martínez-Torregrosa & López-Gay, 2005). We are, therefore, convinced that these results would improve if the teachers of mathematics and physics would teach the same meaning of the fundamental concepts of calculus.

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